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LETTER TO THE EDITOR

A lower bound for the spin glass order parameter of the infinite-ranged Ising spin glass model†

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Abstract. A lower bound for the order parameter of the SK model results from the convergence condition of the TAP equations. The (slightly modified) TAP solution is compatible with this bound. Other published solutions can be excluded.

A satisfactory mean-field theory of spin glasses has not appeared. The simplest model which is expected to show such a behaviour is the model of Sherrington and Kirkpatrick (1975, referred to hereafter as SK) for N Ising spins ($S_i = \pm 1$) interacting via

$$\mathcal{H} = -\tfrac{1}{2} \sum_{i \neq j} J_{ij} S_i S_j \tag{1}$$

with infinite-range exchange interactions. The interactions are independent, but equally distributed according to Gaussian distributions with (in the simplest case) zero means and with standard deviations of $JN^{-1/2}$.

It is widely accepted that the best approach to the solution of the SK model is the treatment of Thouless *et al* (1977, referred to hereafter as TAP). According to this theory the external fields h_i^{ex} can be obtained from

$$\beta h_i^{ex} = \frac{1}{2} \ln \frac{1+m_i}{1-m_i} - \sum_j \beta J_{ij} m_j + m_i \sum_j \beta^2 J_{ij}^2 (1 - m_j^2) \tag{2}$$

where $\beta^{-1} = kT$ and where the local magnetisation is denoted by m_i.

Equation (2) strongly suggests that the external fields are a power expansion up to the second order in the exchange couplings. The author (Plefka 1982) has shown that this is indeed the case and has given the convergence condition for the TAP equations

$$(\beta J)^{-2} > \max \{(1 - 2q_2 + q_4); 2(q_2 - q_4)\} \tag{3}$$

where

$$q_\nu = \frac{1}{N} \sum_i m_i^\nu. \tag{4}$$

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The condition (3) was obtained by an investigation of the singularities of the susceptibility matrix χ_{ij}. The instability $(\beta J)^{-2} = 2(q_2 - q_4)$ results from the term $-2\beta J_{ij}^2 m_i m_j$ in χ_{ij}^{-1} (calculated from (2) by $\partial h_i^{ex}/\partial m_j$) while the other instability corresponds to that found in other treatments (de Almeida and Thouless 1978, Sommers 1978).

It is the aim of this note to present the lower bound for the spin glass order parameter (Edwards and Anderson 1975) resulting from condition (3) and investigate whether the published 'solutions' of the SK model are compatible with this result.

To find this bound let us consider the RHS of condition (3) and let q_2 be fixed but arbitrary. Then the RHS is a function of q_4 which will be denoted by

$$f(q_4) = \begin{cases} 1 - 2q_2 + q_4 & \text{for } \; 3q_4 \geq 4q_2 - 1 \\ 2(q_2 - q_4) & \text{for } \; 3q_4 \leq 4q_2 - 1. \end{cases} \tag{5}$$

As $q_2^2 \leq q_4 \leq q_2$ (the upper bound results from $q_4 = N^{-1} \Sigma\, m_i^4 \leq N^{-1} \Sigma\, m_i^2$ as $m_i^2 \leq 1$) q_4 is restricted to this interval which is indicated in figure 1 where $f(q_4)$ is plotted. Depending on the value of q_2 the function $f(q_4)$ takes the minimal value at the endpoint (figure 1(*a*)) or in the interior of the q_4 interval (figure 1(*b*)). Denoting the minimum of $f(q_4)$ by $f_{\min}$ we find

$$f_{\min} = \begin{cases} f(q_4 = q_2^2) = (1 - q_2)^2 & \text{for } \; 0 \leq q_2 \leq \frac{1}{3} \\ f(q_4 = \frac{4}{3}q_2 - \frac{1}{3}) = \frac{2}{3}(1 - q_2) & \text{for } \; \frac{1}{3} \leq q_2 \leq 1. \end{cases} \tag{6}$$

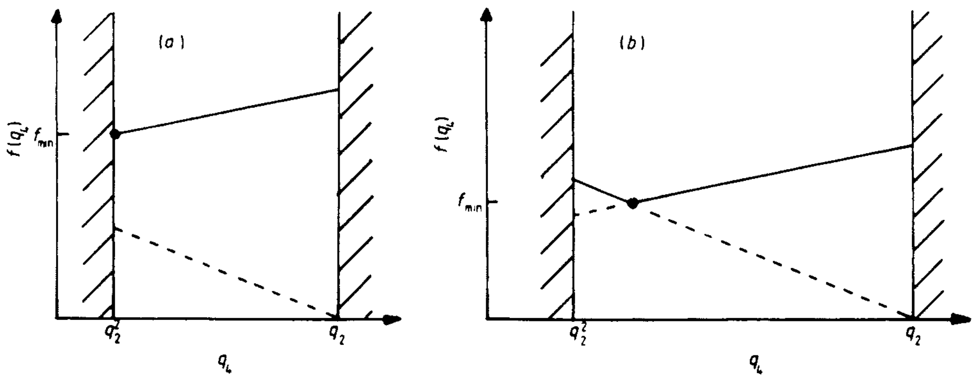

Figure 1. The function $f(q_4)$ defined in (5) against q_4. (*a*), $q_2 < \frac{1}{3}$; (*b*) $q_2 > \frac{1}{3}$.

As $t^2 > f(q_4) \geq f_{\min}$ where $t = (\beta J)^{-1}$ is the reduced temperature we obtain the lower bound for the spin glass order parameter

$$q_2 > \begin{cases} 1 - \frac{3}{2}t^2 & \text{for } \; 0 \leq t \leq \frac{2}{3} \\ 1 - t & \text{for } \; \frac{2}{3} \leq t. \end{cases} \tag{7}$$

This is already the central result of this note and this bound is plotted in figure 2. As $q_2 = 0$ (formally leading to the minimum of the TAP free energy) is not permitted for $t < 1$, our result clearly shows that there is a phase transition in the SK model.

Let us now investigate the main 'solutions' of the SK model for temperatures below the critical temperature ($t \leq 1$).

The TAP solution was given in the critical region ($t \approx 1$) and for low temperatures $t \to 0$. For $t \lesssim 1$ the TAP result $q_2 = 1 - t$ is compatible with (7). The temperature dependence of the $t \to 0$ TAP solution $q_2 = 1 - \alpha t^2$ again is compatible with (7). A

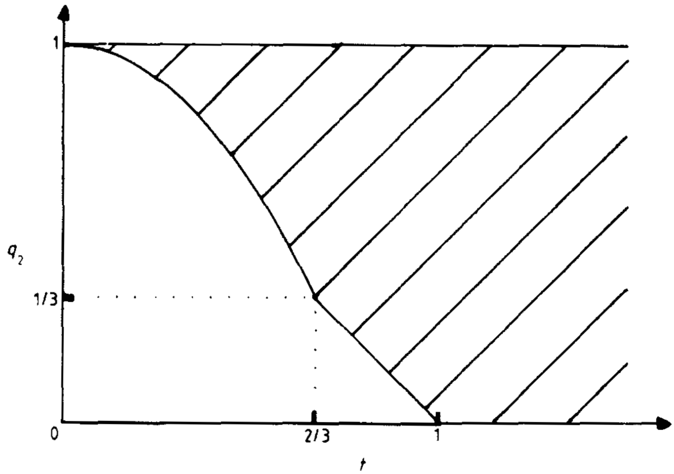

Figure 2. Permitted values of the order parameter q_2 (indicated by the shaded region) in the q_2-t plane where $t=(\beta J)^{-1}$ is the reduced temperature.

disagreement is found for the values of α. TAP claim $\alpha = 1.665$ while our treatment gives $\alpha \leqslant 1.5$. However, using the value of $\alpha = 1.5$ in the TAP solution, all the results of TAP are just slightly modified and still agree—within the error bars—with the results of the computer simulation (Kirkpatrick and Sherrington 1978). Besides this fact we would like to point out that equation (27) of the TAP paper, being quadratic in α, is numerically satisfied for $\alpha = 1.665$ and for $\alpha = 1.5$.

The SK solution obtained first with the replica method certainly violates (7) for low temperatures, but seems at first sight possible for higher temperatures ($\frac{2}{3} \leqslant t < 1$). This is, however, not the case. In the SK solution the internal field is Gaussian distributed (Plefka 1976) implying

$$q_\nu = (2\pi q_2 J^2)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{d}H \tanh^\nu(\beta H) \exp(-H^2/2q_2J^2). \tag{8}$$

With this identification, $t^2 < 1 - 2q_2 + q_4$ holds for the SK solution for all temperatures $t<1$ (de Almeida and Thouless 1978). This is in contradiction to condition (3) and we conclude in agreement with the replica treatment of de Almeida and Thouless (1978) that the SK solution must be rejected for all temperatures $t<1$.

The solution of Sommers (1978) is based on $t^2 < 1 - 2q_2 + q_4$ and thus again is in conflict with condition (3) and has therefore to be rejected too. This conclusion is in agreement with de Dominicis and Garel (1979) and Bray and Moore (1980) obtained, however, from different considerations.

Summing up, we have presented a lower bound for the spin glass order parameter q_2 of the SK model and shown that the TAP solution (slightly modified at low temperatures) is the only one which is compatible with the convergence condition of the TAP equations. It is interesting that the TAP solution and, within the error bars, the computer 'solution' (Kirkpatrick and Sherrington 1978) for q_2 coincides with the lower bound of q_2. This coincidence has been suspected (and seems to be used in the low-temperature region) in the TAP paper and our results support this idea.

At first sight this coincidence seems to be strange, as in this case the value of q_4 is determined by q_2 (see equation (6)). This implies restrictions to the distribution of the m_i. A lot of 'freedom', however, remains for the distribution of the m_i (even $q_4 = q_2^2$, which implies $m_i^2 = q_2$, permits 2^N different m_i distributions). We can imagine that one of these distributions (at least nearly) satisfies the TAP equations (known to have many solutions) for the $h_i^{ex} = 0$ case and gives in addition the minimum of the free energy. However, this is only speculation and further investigations are needed in this direction.

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